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Numerical Evaluation of Fractional Derivatives

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Abstract—*In this paper we describe a simple but effective method for the evaluation of Fractional- order derivatives. This approach is based on the fact that for wide class of Functions, which appear in real physical and engineering application, Riemann-Liouville Derivatives and Grunwald-Letnikov Fractional Derivatives are equivalent. This allows us to use an approximation arising from the Grunwald-Letnikov Fractional Derivatives are equivalent definition for the evaluation of Fractional derivatives of both types. We also Formulate the principal of short memory which reduces the amount of computation.*

Keyword*: Fractional Derivatives, Laplace Transform, Convolution of functions, existence and uniqueness of solution.*

1. INTRODUCTION: FRACTIONAL CALCULUS

Fractional Calculus is a term used for the theory of derivatives and integrals of arbitrary order, which generalize the notion of integer order differentiation and n-fold integration. The idea behind Fractional calculus is to generalize the definition of differentiation and integration with order $n \in \mathbb{N}$ to order $s \in \mathbb{R}$. The first discussion [9] on Fractional Calculus began in 1695 in a letter to L'Hopital by Leibniz in which he discussed about calculus of arbitrary order. Fractional Calculus is three centuries old. Few names that laid the foundation of Fractional Calculus are Abel, Liouville, Riemann, Euler, Caputo etc. Fractional Calculus has recently been applied in various areas of engineering, science, finance, applied mathematics and bio engineering.[10] . It has earlier been observed that derivatives of non-integer order are useful for describing the properties of various real materials like polymer, rocks etc. Also the fractional order models were found more logical to talk an discuss about than the integer-order models. In this paper we are focusing on Fractional Derivatives. Different people gave different definitions for the Fractional Derivative. Few definitions are :

1.1 Grunwald-Letnikov Fractional Derivatives: Let us consider a continuous function f(t), We define

$$ {}_aD_t^p f(t) = \lim_{\substack{h\to 0\\ nh=t-a}} h^{-p} \sum_{r=0}^{n} (-1)^r \binom{p}{r} f(t-rh) $$

$$ {}_aD_t^\alpha f(t) = \lim_{h\to 0} \frac{{}_a\Delta_h^\alpha f(t)}{h^\alpha},\ {}_a\Delta_h^\alpha f(t) = h^{-p} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh) \qquad (1.1) $$

Where [x] means the integer part of x. The above formula has been obtained under the assumption that the derivatives $f^{(k)}(t)$ (k=1, 2, 3,. . . ,m+1) are continuous in the closed interval [a,t] and that m is the integer number satisfying m > p-1.

1.2. Riemann-Liouville Derivatives:

$$ {}_aD_t^p f(t) = \left(\frac{d}{dt}\right)^{m+1} \int_a^t (t-\tau)^{m-p} f(\tau)d\tau, (m \le p < m+1) $$

Closely related to fractional –order differentiation is fractional-order integration:

$$ {}_aD_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\tau)}{(t-\tau)^{(1-\alpha)}} d\tau \quad \alpha > 0 $$

It is worth noted that

$$ {}_0D_t^\alpha ({}_0D_t^{-\alpha} f(t)) = f(t)\ \alpha > 0 $$

Which generalizes an analogous property of integer derivatives and integrals.

1.3. Caputo's Fractional Derivatives

The definition of the fractional differentiation of the Riemann-

Liouville Derivatives type played an important role in the development of the theory of fractional derivatives and for its applications in pure mathematics. However, the demands of modern technology require a certain revision of well established mathematical approach .The Caputo approach provides an interpolation between an integer order derivatives:

$$ {}^CD^\alpha f(x) = \frac{1}{\Gamma(\alpha-n)} \int_a^x \frac{f^{(n)}(u)}{(x-u)^{(\alpha-n+1)}}\ ,\ n-1 < \alpha < n, \alpha \in \mathbb{R}, n \in \mathbb{N} $$

1.4 Euler's Fractional Derivatives:

$$ \frac{d^\alpha}{dt^\alpha}[t^\beta] = D_t^\alpha[t^\beta] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} t^{\beta-\alpha}, \alpha \in \mathbb{R} $$

1.5. Sequential Fractional Derivatives

The main idea of differentiation and integration of arbitrary order is the generalization of iterated integration and differentiation. In all these approaches we replace the integer valued parameter n of a operator denoted by $\frac{d^n}{dt^n}$ with a non integer parameter p.

However, we can assume that the n-th order differentiation is simply a series of n first order differentiation .So, considering more general expressions

$$D_t^{\alpha} = D_t^{\alpha_1} D_t^{\alpha_2} D_t^{\alpha_3} \dots\dots\dots . D_t^{\alpha_n}$$

Where $\alpha = \alpha_1 + \alpha_2 + \alpha_3 + \cdots \dots\dots . \alpha_n$,which we will also call the sequential fractional derivatives.

Indeed, Riemann-Liouville Derivatives can be written as

$${}_aD_t^p f(t) = \frac{d}{dt}\frac{d}{dt} \dots\dots . \frac{d}{dt}\, {}_aD_t^{-(n-p)} f(t) \quad (\text{n-1} \le p < n)$$

While the Caputo fractional differential operator can be written as

$${}^{C}D^{\alpha} f(x) = {}_aD_t^{-(n-p)} \frac{d}{dt} \dots\dots . \frac{d}{dt} \text{f(t)} (\text{n-1} < p \le n-1)$$

2. PROPERTIES OF FRACTIONAL DERIVATIVES:

Fractional Derivatives satisfy almost all the properties that hold for[5] ordinary derivatives. We are aware of the general properties of the derivative operator $D_t^n, n \in \mathbb{N}$. Below mentioned are the properties of Fractional Derivative that can be easily verified:

- $D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$
  where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.
- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$.
- $D_t^{\propto}[h(t) + g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t) + g(t)] = D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)]\, , x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(\text{m})} t^{-(m+\propto)}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$.

$D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$,

where $\alpha \in D_t^{\propto}[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[g(t)]$,

where $\binom{\propto}{k} = \frac{\Gamma(\propto+1)}{\Gamma(k+1)\Gamma(\propto+1-k)}$.

- $D_t^{\propto}[f(t)C] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[f(t)] D_t^k[C] = D_t^{\propto}[f(t)]C$
  Where C is an arbitrary constant.
- $D_t^{\propto}[h(t) + g(t)] = \sum_{k=0}^{\infty}\binom{\propto}{k} D_t^{\propto-k}[t^0] D_t^k[h(t) + g(t)]$
  $= D_t^{\propto}[h(t)] + D_t^{\propto}[g(t)]$.
- $D_t^{\propto}[h(at)] = a^{\propto} D_x^{\propto}[h(x)]$ under the scaling $x = at$.
- $D_t^{\propto}[t^{-m}] = (-1)^{\propto} \frac{\Gamma(m+\propto)}{\Gamma(\text{m})} t^{-(m+\propto)}$ for a given $m \in \mathbb{R}$.
- $D_t^{\mu+\nu}[f(t)] = D_t^{\mu}[D_t^{\nu}(f(t))] = D_t^{\nu}[D_t^{\mu}(f(t))]$ under the composition of D_t^{ν} and D_t^{μ}on $f(t)$.
- $D_t^{-1}[t^{\beta}] = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+1)} t^{\beta+1} = \frac{t^{\beta+1}}{\beta+1}$, where $\beta \in \mathbb{R}$ corresponding to a negative order derivative.

2.1. Mittag-Leffler Function:

The Exponential function play a important role in the theory of integer order differential equation its one parameter generalization is denoted by[4]

$$\boldsymbol{E_{\alpha}(z)} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$$

was introduced by G.M Mittag Leffler [5, 6, 7] and also studied by A.William[8, 9] **.**

2.2 Laplace Transforms of Fractional Derivatives:

The Laplace transform of a function f(t) is defined as

$\text{F(s)} = \text{L(f(t))} = \int_0^{\infty} e^{-st} \text{f(t) dt}$

Laplace Transform of Fractional derivatives of order $p > 0$ in terms of Riemann-Liouville Derivatives p[th]

$$\text{L}\{{}_0D_t^p f(t); s\} = s^p \text{F(s)} - \sum_{k=0}^{n-1} s^k \left[{}_0D_t^{p-k-1} f(t)\right]_{t=0} \quad (\text{n-1} \le p < n)$$

3. APPROXIMATION OF FRACTIONAL DERIVATIVES

We use the following approximation, arising from Grunwald-Letnikov definition

$${}_aD_t^{\alpha} f(t) \approx {}_a\Delta_t^{\alpha} f(t) \qquad (3.1)$$

Another type of approximation can be obtained from the Riemann-Liouville definition by n-times integration by parts and subsequent approximation of integral containing $f^n(\tau)$.

3.1 The short Memory Principle

For $t \ge a$ the number of addends in the fractional derivative approximation (3.1) become enormously. However it follows drom(i.i) that for a large t the role of the history of the behavior of the function f(t) near the lower terminal can be neglected under certain assumption. Those observations leads us to the formulation of the short memory principle, which means taking in to account the behavior of f(t) only in the interval [t-L,t],where L is the memory length

$${}_aD_t^{\alpha} f(t) \approx {}_{t-L}D_t^{\alpha} f(t) \quad (t > a + L) \qquad (3.1.1)$$

which means the fractional derivative with lower limit a is approximated by fractional derivatives with moving lower limit t-L.Due to this approximation, the number of addends in approximation(3.1) can not be greater than$\left[\frac{L}{h}\right]$.

If$f(t) \leq M$ for $a \leq t \leq b$,we easily establishthe following estimate for the error caused due to short memory Principle:

$$\Delta(t) = |{}_aD_t^\alpha f(t) - {}_{t-L}D_t^\alpha f(t)| \leq \frac{ML^{-\alpha}}{|\Gamma(i-\alpha)|} \qquad a+L \leq t \leq b \quad (3.1.2)$$

The above inequality can be used to find the memory length L providing the required accuracyϵ:

$$\Delta(t) \leq \epsilon, \quad a+L \leq t \leq b \text{ , if } L \geq \left(\frac{M}{|\Gamma(i-\alpha)|\epsilon}\right)^{1/\alpha} \qquad (3.1.3)$$

3.2. Order of Approximation

Let us recall some basic facts on the approximation of integer-order derivatives.

It is well known that backward finite differences can be used for approximating integer order derivatives. For example, for a fixed t and a small step h we can approximate the first order derivative by two point backward difference:

$$y'(t) \approx \widetilde{y'(t)} = \frac{y(t)-y(t-h)}{h} \qquad (3.2.1)$$

Writing y(t-h) in the form of the Taylor's series, we have

$$\widetilde{y'(t)} = \frac{y(t)-y(t-h)}{h} = y'(t) - \frac{y''(t)}{2}h + \cdots..$$
$$= y'(t) + O(h),$$

Which means that $y(t) - \widetilde{y'(t)} = O(h)$; (3.2.2)

In other words, the two-point backward difference formula gives the first order approximation of$y'(t)$.

In this case, we can write the approximation of the α-th derivative as

$$ {}_0\widetilde{D_t^\alpha} f(t) = h^{-\alpha} \sum_{j=0}^n (-1)^j \binom{\alpha}{j} f(t-jh) \qquad (3.2.3)$$

$$= h^{-\alpha} \sum_{j=0}^n \binom{j-\alpha-1}{j} f(t-jh). \qquad (3.2.4)$$

To introduce the idea of the considerations which will follow, let us take the simplest function

$f_0(t) \equiv 1 \ (t \geq 0)$. we already know that its exact α-th derivative is ${}_0D_t^\alpha f_0(t) = \frac{t^{-\propto}}{\Gamma(1-\alpha)}$.

On the other hand, the approximation (3.1.4) gives the approximate value

$$ {}_0\widetilde{D_t^\alpha} f_0(t) = h^{-\alpha} \sum_{j=0}^n \binom{j-\alpha-1}{j}$$

Using the known summation formula for the binomial coefficients

$$\sum_{j=0}^n \binom{j-\alpha-1}{j} = \binom{n-\alpha}{n} \qquad (3.2.5)$$

Let us now consider $f_m(t) = t^m$, m = 1, 2, 3…….. .

In this case the exact α-th derivative is

$$ {}_0D_t^\alpha f_m(t) = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)} t^{m-\alpha}$$

And the approximation of the exact derivative becomes

$$ {}_0\widetilde{D_t^\alpha} f_m(t) = t^{m-\alpha} n^\alpha \sum_{j=0}^n \binom{j-\alpha-1}{j} \left(1-\frac{j}{n}\right)^m, \qquad (3.2.6)$$

Or, after expanding the binomial,

$$ {}_0\widetilde{D_t^\alpha} f_m(t) = t^{m-\alpha} \sum_{r=0}^m (-1)^r \binom{m}{r} n^{\alpha-r} \sum_{j=0}^n \binom{j-\alpha-1}{j} (j)^r \quad (3.2.7)$$

The sum

$$S = \sum_{j=0}^n \binom{j-\alpha-1}{j} (j)^r = \sum_{i=1}^r \sigma_r^{(i)} \frac{\Gamma(n-\alpha+1)}{(i-\alpha)\Gamma(-\alpha)\Gamma(n-i+1)}. \qquad (3.2.8)$$

On substituting

$$ {}_0\widetilde{D_t^\alpha} f_m(t) = \frac{t^{m-\alpha}}{\Gamma(-\alpha)} \sum_{r=0}^m (-1)^r \binom{m}{r} \sum_{i=1}^r \sigma_r^{(i)} \frac{n^{\alpha-r}\Gamma(n-\alpha+1)}{(i-\alpha)\Gamma(n-i+1)} .(3.2.9)$$

3.3 Computation of coefficients

For implementing the fractional difference method of computation of fractional derivative, it is necessary to compute the coefficient

$w_k^{(\propto)} = (-1)^k \binom{\propto}{k}$

where k=1,2,3……………..

Where ∝ is the order of fractional order derivatives.

For fixed value of ∝, we have

$w_0^{(\propto)} = (-1)^k$; $w_k^{(\propto)} = (1 - \frac{\alpha+1}{k}) w_{k-1}^{(\propto)}$ k=1,2,3…………

However in some problem $w_k^{(\propto)}$ can be expressed in Fourier Transform:

$w_k^{(\propto)} = \frac{1}{2\pi i} \int_0^{2\pi} f_\alpha(t) e^{-tik} dt$, $f_\alpha(t) = (1 - e^{-ti})^\alpha$

4. CONCLUSION

We have described the method for the numarical evaluation of Fractional- order derivatives. This approach is based on the fact that for wide class of Functions, which appear in real physical and engineering application, Riemann-Liouville Derivatives and Grunwald-Letnikov Fractional Derivatives are equivalent. This allows us to use an approximation arising from the Grunwald-Letnikov Fractional Derivatives are equivalent definition for the evaluation of Fractional derivatives of both types. We also Formulate the principal of short memory which reduces the amount of computation and describe method of computing coefficients $w_k^{(\propto)} = (-1)^k \binom{\propto}{k}$ for calculating fractional derivative to implement difference method.

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